Fano 4-folds with $b_2 > 12$ are products of surfaces

- Smooth, complex Fano 4-folds

  X smooth Fano variety

  - classification up to dim 3

    del Pezzo surfaces $\rightsquigarrow$ $\rho_X \le 9$

    dim 3: 105 families '80s

- finitely many families in each dimension
- Picard number: $\rho_X = b_2(X)$

Thm (Mori–Mukai '86). Let X be a Fano 3-fold. If $\rho_X > 5$, then $X \cong S \times \mathbb{P}^1$, S a del Pezzo surface.

$$\rho_{S \times \mathbb{P}^1} = 1 + \rho_S \le 10$$

$\Rightarrow$ $\forall X$ Fano 3-fold $\rho_X \le 10$

& : for $\rho = 6, .., 10$ only $S \times \mathbb{P}^1$.

Thm (C. '23) Let X be a Fano 4-fold. If $\rho_X > 12$, then $X \cong S_1 \times S_2$, S_i del Pezzo.

$$\rho_{S_1 \times S_2} = \rho_{S_1} + \rho_{S_2} \le 18$$

$\Rightarrow$ $\forall X$ Fano 4-fold $\rho_X \le 18$

& : for $\rho = 13, .., 18$: only $S_1 \times S_2$.

- all known examples of Fano 4-folds not products have $\boxed{\rho \le 9}$

Work in progress: also $\rho = 12$ only $S_1 \times S_2$

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- very few examples (not products) for $\rho \geq 6$:

$\rho=6$: 10 known families (7 toric)

$\rho=7,8,9$: 1 known family in each ρ.

X a Fano 4-fold

A CONTRACTION of X is a surjective morphism $f: X \to Y$, connected fibres, Y normal & proj.

f is ELEMENTARY if $\rho_X - \rho_Y = 1$

$$\left. N_1(X) = \text{1-cycles}, \mathbb{R}\text{-coeff.} / \text{num. equiv.} \right] \dim \rho_X$$

$\cup$

$NE(X)$ cone of effective curves
rational polyhedral cone of dim. ρ_X

There is a bijection: EXTREMAL RAYS
$\parallel$

$$\left\{ \text{elementary contractions of } X \right\} \longleftrightarrow \left\{ \text{1-dim. faces of } NE(X) \right\}$$

$$f \longmapsto NE(f)$$

$$\cap \qquad\qquad \cap$$

$$\left\{ \text{contractions of } X \right\} \longleftrightarrow \left\{ \text{faces of } NE(X) \right\}$$

<u>Thm 1</u> (C. '22) Let X be a Fano 4-fold. If X has a small elementary contraction, then $\rho_X \leq 12$.

$i: D \hookrightarrow X$ a prime divisor

$$i_*: N_1(D) \longrightarrow N_1(X)$$
$$[C] \longmapsto [C]$$

$N_1(D,X) := i_*(N_1(D)) \subseteq N_1(X)$

$\mathcal{N}_1(D,X) := i_*(\mathcal{N}_1(D)) \subseteq \mathcal{N}_1(X)$
↳ linear span in $\mathcal{N}_1(X)$ of classes of curves in D

$$\dim \mathcal{N}_1(D,X) \leq \rho_D$$

Thm 2 (C. '12, C.-Romano '22, C.-Romano-Secci '22).
Fano 4-folds s.t. $\exists D \subset X$ with $\dim \mathcal{N}_1(D,X) \leq \rho_X - 3$ are classified. Either $X \cong S_1 \times S_2$, or $\rho_X \in \{5,6\}$ (→ 17 families).

⇝ we can assume that X has the property:
(∗) $\forall D \subset X$ prime divisor, $\rho_D \geq \dim \mathcal{N}_1(D,X) \geq \rho_X - 2$

Thm 3 If X satisfies (∗), and has no small elementary contraction, then $\rho_X \leq 12$.

Consequences of (∗) on contractions of X:
let X with (∗) and no small elem. contr.

Fact 1: if $g: X \to Z$ is a contraction of fiber type, then $\rho_Z \leq 4$.

Fact 2: either $\rho_X \leq 5$, or: every elementary contraction of X is "OF TYPE (3,2)"
i.e: $f: X \to Y$ is birational, divisorial $E = \mathrm{Exc}(f)$
& $S := f(E) \subset Y$ has dim. 2

$$\begin{array}{ccc} f: X & \longrightarrow & Y \\ \cup & & \cup \\ E & & S \end{array}$$

type (3,2)
$C \subset E$ a general fiber
$-K_X \cdot C = 1$ $\boxed{E \cdot C = -1}$

Note: if $\dim \mathcal{N}_1(E,X) \leq 3$, then $\rho_X \leq 5$ by (∗)

$\rho_X \leq 5$ by $(\ast)$

$\Rightarrow$ we can assume that:

$\forall f \quad \rho_E \geq \dim N_1(E, X) \geq 4$

This implies that: • Y is Fano too

• different elementary contractions have different exceptional divisors

$\boxed{\text{Geometry of } f}$ (Andreatta – Wiśniewski '90 s)

$$\begin{array}{ccc} f: X & \longrightarrow & Y \\ \cup & & \cup \\ E & & S \end{array}$$

• f may have isolated 2-dim. fibers F

If $y = f(F) \in S$, then Y and/or S are singular at y

(possible singularities are classified)

• Y has isolated, loc. fact, terminal sing.

• S can be <u>not normal</u>

Outside the 2-dim. fibers & their images:

Y and S are smooth

f is just the blow-up of S

E $\mathbb{P}^1$-bdle over S

Simplifying assumption: <u>no 2-dim. fibers</u>

Y is a smooth Fano 4-fold

S is a smooth surface

E is a smooth $\mathbb{P}^1$-bdle $/S$.

Strategy: prove that <u>S is del Pezzo</u>

$\Rightarrow \rho_S \leq 9 \Rightarrow \rho_E \leq 10 \Rightarrow \rho_X \leq 12$.

In fact we show that:

The fact we show that:

$$-K_S = \underbrace{(-K_Y)}_{\text{ample}}|_S$$

Set $L := (-K_Y)|_S$ ample on S

consider: $K_S + L$

- $K_S + L$ is <u>nef</u>

Recall: $\rho_E \geq 4 \Rightarrow \rho_S \geq 3$

$$\overline{NE}(S) = \underbrace{\overline{NE}(S)_{K_S \geq 0}}_{\text{here}} + \sum_{\substack{R_i \text{ extr. rays} \\ (K\text{-negative})}} R_i$$

$K_S + L > 0$

R_i extr. ray of S

- $S \to pt$ <u>NO</u> $\rho_S \geq 3$
- $S \xrightarrow{\mathbb{P}^1\text{-bdle}} C$ <u>NO</u> $\rho_S \geq 3$
- contraction of (-1)-curve $\Gamma \subset S$

$(K_S + L) \cdot \Gamma = L \cdot \Gamma - 1 \geq 0$

$\Rightarrow K_S + L$ semiample $\rightsquigarrow$ it defines a contraction $\varphi : S \longrightarrow T$ K-negative ($-K_S$ φ-ample)

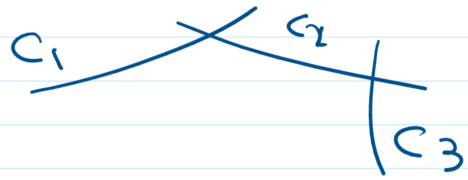
φ contracts curves s.t. $(K_S + L) \cdot C = 0$ $L = (-K_Y)|_S$

Goal: $-K_S = (-K_Y)|_S$ i.e. $K_S + L \equiv 0$

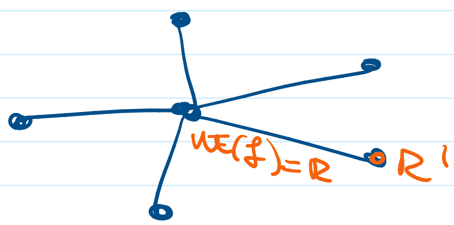
φ
- blowup of distinct smooth pts on a smooth surface
- a conic bdle onto a smooth curve
- $\boxed{\text{contraction to a pt} \rightsquigarrow K_S + L \equiv 0}$

contraction to a pt ⇝ $K_S + L = 0$

To exclude the first two cases: we construct 3 irreducible curves C_1, C_2, C_3 distinct contracted by φ intersecting each other



To construct these curves:
consider $NE(f) \subset NE(X)$ extr. ray
& consider the 2-dimensional faces of $NE(X)$ containing $NE(f)$



$R + R'$ 2-dim. face

$\updownarrow$

$g : X \to Z$ contraction of X
$\rho_X - \rho_Z = 2$

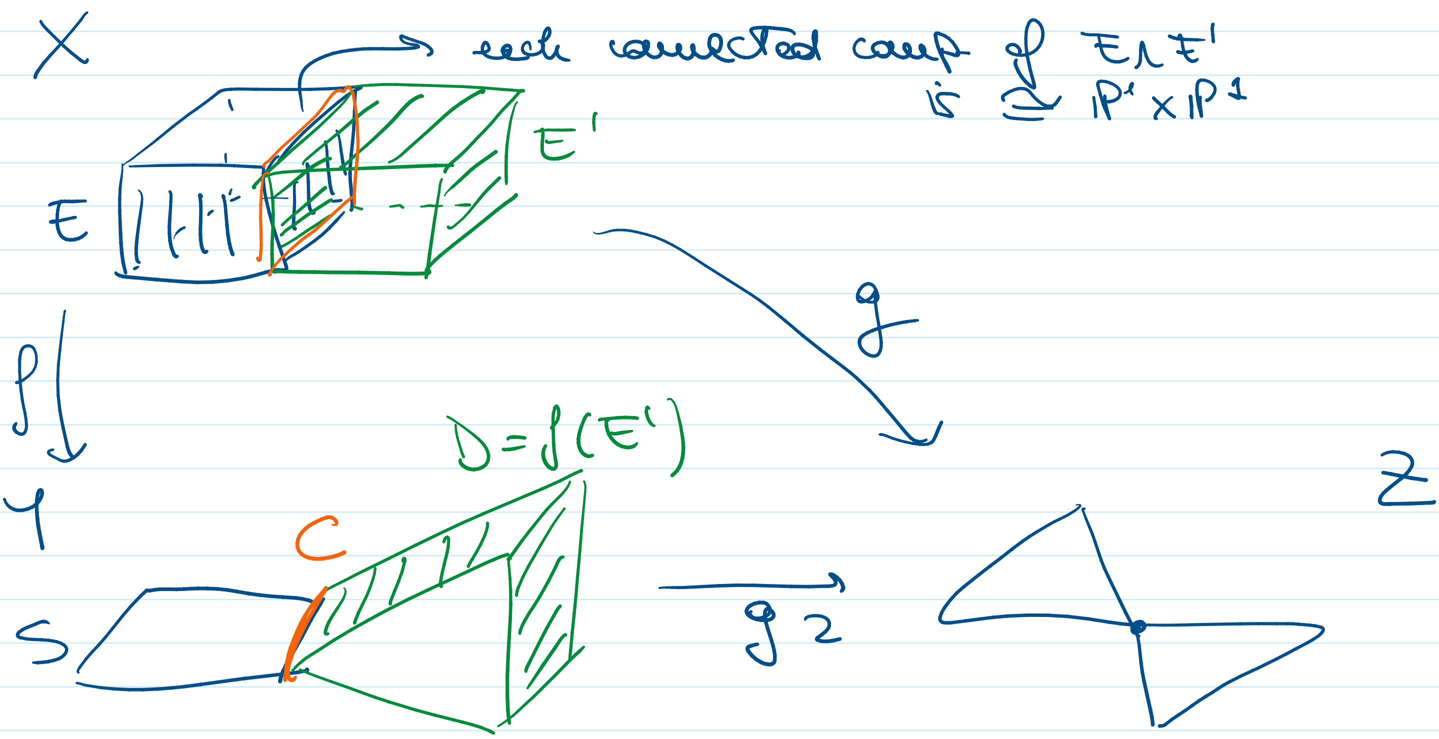
If g is of fiber type: $\rho_Z \leq 4 \Rightarrow \rho_X \leq 6$
we can assume that $\forall\, R + R'$, g is birational.
R' also gives a contraction of type $(3,2)$ with exceptional divisor E'

Fact: since $\dim N_1(E, X) \geq \rho_X - 2$ by $(*)$
E can be disjoint from at most 2 other exceptional divisors E'
⇝ we can choose $R + R'$ s.t. E' intersects E

$\Rightarrow$ E & E' must intersect in this way.

$\Rightarrow$ E & E' must intersect in this way:

$$E\cdot R' = E'\cdot R = 0$$



- g_2 is again the blow-up of a smooth surface
- g blows-up two surfaces intersecting in pts

C is a fiber of g_2 $\Rightarrow$ $C\cong\mathbb{P}^1$
$-K_Y\cdot C = 1$

C is a (-1)-curve in S $\Rightarrow$ $-K_S\cdot C = 1$

$\Rightarrow$ $(K_S+L)\cdot C = 0$, $\varphi(C)=pt.$

$\rho_X\geq 8 \rightsquigarrow$ get C_1, C_2, C_3 in this way.

$\rho = 7,8,9$: $\quad \hat{X} \xrightarrow[\text{- - -}]{\text{cont flips}} X$ smooth Fano 4-fold

$\hat{X} \xrightarrow{\text{blowup general pts } (6,7,8)} \mathbb{P}^4$

$\rho = 9$ Mukai

C., Codogni, Fanelli